Anomalous U(1) Vortices and The Dilaton

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The role of the (dynamical) dilaton in the vortices associated with the spontaneous breaking of an anomalous U(1) from heterotic string theory is examined. We demonstrate how the anomaly (and the coupling to the dilaton-axion) can appear in the Lagrangian and associated field equations as a controlled perturbation about the standard Nielsen-Olesen equations. In such a picture, the additional field equation for the dilaton becomes a series of corrections to a constant dilaton VEV as the anomaly is turned on. In particular we find that even the first nontrivial correction to a constant dilaton *generically* leads to a (positive) logarithmic divergence of the heterotic dilaton near the vortex core. Since the dilaton field governs the strength of quantum fluctuations in string theory, this runaway behavior implies that anomalous U(1) vortices in string theory are intrinsically quantum mechanical objects.

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I. INTRODUCTION

Many four-dimensional compactifications of superstring theory [1–3] which preserve an unbroken N=1 spacetime supersymmetry also possess a U(1) gauge symmetry with apparently anomalous content for the massless fermions of the associated gauge charge. The apparent anomalies of these U(1) gauge groups are canceled by a four-dimensional remnant of the Green-Schwarz mechanism [4], as originally argued by Dine, Seiberg, and Witten [5–7].

These authors noted that while the superpotential is not renormalized in either string or sigma model perturbation theory (so that solutions of the string equations at lowest order remain solutions to all orders and the vacuum remains perturbatively stable), vacuum degeneracy can still be lifted if a compactifcation contains a gauge group with an unbroken U(1) subgroup, by generating a Fayet-Iliopoulos [8] D-term. By assumption such a term is not present at the tree level in the loop or sigma-model expansion, so the question arises as to whether it is possible to generate it radiatively in perturbation theory. It turns out that it can arise only at one loop in the string-loop expansion, and then only if the U(1) is anomalous (since the term is proportional to the trace over the U(1) charges of the left-handed massless fermions [6]).

In fact many string compactifications have precisely such an anomalous U(1), with an explicit example being furnished by Dine, Seiberg and Witten for the SO(32) heterotic string. They argue that the anomalies induced by such a U(1) are canceled by assigning the model-independent axion a nontrivial U(1) gauge variation, corresponding to the remnant of the underlying ten-dimensional Green-Schwarz anomaly cancellation mechanism. Supersymmetrically, the model-independent axion is paired with the dilaton [whose vacuum expectation value (VEV) sets the string-loop coupling constant] to form the scalar component of a chiral multiplet, whose modified (due to the anomaly cancellation and gauge invariance) Kahler potential now yields the Fayet-Iliopoulos term. The effect of this induced Fayet-Iliopoulos D-term, generically, is to break spacetime supersymmetry as a one-loop effect in the string loop expansion. However, the full D-term also includes contributions from charged scalars in the theory. In the known cases some of these scalars can acquire VEVs to cancel the Fayet-Iliopoulos D-term thereby restoring supersymmetry by spontaneously breaking the U(1) symmetry in a process referred to as vacuum restabilization.

It has recently been argued that in heterotic $E_8 \times E_8$ (as opposed to heterotic $SO(32)$) compactifications, the axion involved in the anomaly cancellation is a model-dependent axion originating from internal modes of the Kalb-Ramond form field B_{ij}, with $i, j = 4, \ldots, 9$. (The essence of this argument dates back to Distler and Greene [9].) Such axionic modes appear paired with an internal Kahler form zero mode to form the scalar components of complex moduli T_i, which describe the size and shape of the compactification manifold. However as Dine, Seiberg, and Witten had noted [5], if we assign one of the model-dependent axions a nontrivial gauge transformation to cancel the anomaly, and then proceed as in the model-independent case, we again get mass and tadpole terms that now appear at the string *tree* level because there is no longer the dilaton (and hence string-loop) dependence that occurs in the model-independent case. These terms are by assumption absent in the classical, massless limit of string theory. The other way of saying this

[9] is that the U(1) is not a symmetry of the world-sheet construction, and hence is not a symmetry of the low-energy effective theory describing the (classical) string vacuum. Furthermore, there is no Fayet-Iliopoulos term generated in this case, so spacetime supersymmetry is not spontaneously broken and the vacuum destabilized. Thus, henceforth, we will work within the usual framework of Dine, Seiberg, and Witten [5] and consider anomaly cancellation via the dilaton and model-independent axion, or S multiplet.

On the other hand, it is well known that the breaking of a U(1) symmetry can give rise to topological defects known as Nielsen-Olesen vortices [10], which may appear in a cosmological context as cosmic strings [11]. Binétruy, Deffayet, and Peter [12] analyzed the vortices arising from such anomalous U(1) scenarios and concluded that there exist configurations of the axion such that some of these vortices can be local gauge strings, whereas for other choices of the axion configuration the vortices are global [11]. However, in order to arrive at their final model, they freeze the dilaton to its (asymptotic) VEV while leaving the axion dynamical. Since the dilaton and model-independent axion form the scalar component of a chiral superfield, this Ansatz explicitly breaks supersymmetry as they acknowledge. Since vacuum restabilization perturbatively restores supersymmetry in the resulting low-energy effective theory, an analysis of the vortex solutions of this effective theory should retain the fields required by the supersymmetry. In this paper we present such an analysis, and examine the structure of the anomalous U(1) vortex including the dilaton as a dynamical field.

In order to treat the dilaton, axion, and anomaly in a systematic way, we show that the anomaly can be treated in the low-energy effective Lagrangian, and in the field equations, as a perturbation about the Abelian Higgs model and Nielsen-Olesen equations respectively. The dimensionless Green-Schwarz coefficient δ_{gs} will be considered as the perturbation parameter; in the simplified model of [12], wherein a single scalar accomplishes the vacuum restablization, supersymmetry (SUSY) restoration, and U(1) breaking, this parameter is of order 10^{-3}. Then, looking for static, axially symmetric (vortex) solutions of the field equations using the standard Ansatz for the Higgs (scalar) and gauge fields, we show that the axion is only θ dependent (as [12] obtain) and the dilaton is only r dependent given the assumed time-independent, cylindrical symmetry of the fields. The axion field equation effectively decouples (we still obtain the asymptotically converging solution of [12] for the axion, plus the others corresponding to global axionic strings), and we obtain ordinary differential equations (ODEs) for the dilaton, Higgs modulus, and the nontrivial component of the gauge field.

Corrections to a constant dilaton appear only at $O(\delta_{gs})$; at zeroth order we simply obtain the usual Nielsen-Olesen equations for the Higgs and gauge field. Using a parametrization for the solutions to the Nielsen-Olesen equations correct at the asymptotic limits $r \to \infty$, and $r \to 0$, we obtain the first order correction to the dilaton. We find that the correction necessarily diverges logarithmically to positive infinity as $r \to 0$ as a direct consequence of the $r \to \infty$ boundary condition and the two-dimensional nature of the problem. We also show this is not an artifact of the parametrization of the Nielsen-Olesen solutions, but is only dependent on these asymptotic regimes. This divergence reflects a transition to a (heterotic) strong-coupling regime and hence a failure of the effective theory as a classical limit (since the large dilaton field means large quantum effects). Finally, to check the consistency of this result outside of δ_{gs} perturbation theory, we examine exact solutions to the large-dilaton limit of the full dilaton field equation, which involves exponential dilaton self-couplings, and the axion contribution, neither of which is visible in the first order δ_{gs} perturbation theory. We find the same singularity structure of the dilaton at $r = 0$ as the $O(\delta_{gs})$ result, indicating a breakdown of the full classical approximation in the vortex core.

II. MODEL LAGRANGIAN

Independently of the compactification scheme to four dimensions, the antisymmetric tensor field B_{MN} yields via dualization the universal or model-independent axion a, which combines with the four-dimensional dilaton to form the scalar component of a chiral superfield denoted by S. After dimensional reduction to four dimensions, Weyl transformation to Einstein metric, and Poincaré duality, the relevant bosonic terms of the effective action are [2]

$$S_{4D,het} = \int d^4x \; (-G_{(4)})^{\frac{1}{2}} \; \left\{ \frac{1}{2\kappa_4^2} \left[R^{(4)} - 2\, \frac{\partial_\mu S \partial^\mu S^*}{(S+S^*)^2} \right] - \frac{1}{4g_4^2} \; \left[e^{-2\Phi_4} \; F^{a\mu\nu} F^a_{\mu\nu} - a \; F^{a\mu\nu} \tilde{F}^a_{\mu\nu} \right] \right\} + \dots \tag{2.1}$$

where the ellipsis represents compactification-dependent terms involving the other T-like moduli of the orbifold or Calabi-Yau manifold, threshold corrections, and the scalars (matter fields) coming from the ten-dimensional gauge fields. Here, $g_4^2 = \kappa_4^2/\alpha'$, and $S = e^{-2\Phi_4} + ia$ defines the four-dimensional dilaton and (model-independent) axion.

With respect to the general supergravity action [2], the relevant features are the dilaton-axion Kahler potential given by $-\log(S+S^\dagger)$, and the gauge kinetic function given by $f_{ab} = \frac{\delta_{ab}}{g_4^2} S$.

Many compactifications of string theory possess gauge groups containing U(1) subgroups. Sometimes the quantum numbers of the massless fermions associated with such a compactifaction appear to lie in anomalous representations, and hence the U(1) is referred to as anomalous. As Dine, Seiberg, and Witten [5] showed, the Green-Schwarz mechanism of the underlying string theories (which ensures that the string theories themselves are anomaly free) has a four-dimensional remnant which cancels the would-be anomalies associated with U(1). Specifically, the axion-gauge coupling in Eq. (2.1) implies that an anomalous U(1) variation, $\delta\mathcal{L}_{eff} = -\frac{1}{2}\delta_{gs}\lambda F_{\mu\nu}\tilde{F}^{\mu\nu}$ (where δ_{gs} is the anomaly coefficient), can be canceled by assigning the axion a nontrivial U(1) variation: $a \to a + 2\delta_{gs}\lambda$. In terms of the superfield S this reads:

$$S \to S + 2i\delta_{gs}\Lambda, \tag{2.2}$$

where Λ is the supersymmetric generalization of the gauge transformation parameter λ. Gauge invariance implies we must modify the dilaton-axion Kahler potential to

$$K = -\log(S + S^\dagger - 4\delta_{gs}V). \tag{2.3}$$

with $V \to V + i(\Lambda - \Lambda^\dagger)/2$ the vector superfield containing A. Among other terms this induces a one-loop (in the string loop expansion) Fayet-Iliopoulos term [8]. Specializing to the anomalous U(1) sector of the theory, and including the contributions coming from the (other) scalars charged under the U(1), denoting the 4D dilaton now by $\Phi_4 \to \Psi$, and the scalar (chiral) superfields by $\mathcal{A}_i$ with charges X_i and scalar components Φ_i, we can write the effective Lagrangian of our model:

$$\mathcal{L} = \int d^4\theta \left[K(S,S^\dagger) + \mathcal{A}_i^\dagger e^{X_i V}\mathcal{A}_i\right] + \int d^2\theta \frac{1}{4} S W^\alpha W_\alpha + h.c. \tag{2.4}$$

with W^α the spinor (chiral) superfield associated with the field strength of V. While a superpotential for the $\mathcal{A}_i$ could be added, since it must be independent of the dilaton superfield S in perturbation theory, we neglect it for simplicity since we are primarily interested in dilaton-axion dynamics.

Expanding this in component form and eliminating the auxillary field of V by its algebraic equation of motion yields

$$\begin{aligned}\mathcal{L}_{bos} = &-\partial_\mu\Psi\partial^\mu\Psi - \frac{e^{4\Psi}}{4}\left(\partial^\mu a - 2\delta_{gs}A^\mu\right)^2 - (D_\mu\Phi_i)^\dagger D^\mu\Phi_i \\ &-\frac{1}{4}e^{-2\Psi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4}aF^{\mu\nu}\tilde{F}_{\mu\nu} - \frac{e^{2\Psi}}{2}\left(e^{2\Psi}\delta_{gs} + X_i\Phi_i^\dagger\Phi_i\right)^2.\end{aligned} \tag{2.5}$$

Equation (2.5), with the Planck mass restored everywhere (which we have implicitly suppressed by setting $\kappa_4 = \alpha' = 1$) and with s instead of $e^{-2\Psi}$ for the dilaton, agrees with the Lagrangian of reference [12].

III. PERTURBATION SCHEME AND FIELD EQUATIONS

In string theory the dilaton is the string loop expansion parameter, its vacuum expectation value setting the string coupling constant [1]. As is evident from Eq. (2.5), its four dimensional remnant in this model manifestly sets the U(1) *gauge* coupling: $\langle e^\Psi\rangle = g$. Since our main interest is in the dilaton, it will be convenient for our purposes to consider variations of the dilaton about its vev. Thus define $\psi \equiv \Psi - \langle\Psi\rangle$ so that

$$e^\Psi \equiv g e^\psi. \tag{3.1}$$

We will henceforth refer to ψ as the dilaton. Then $\psi = 0 \;\leftrightarrow\; \langle Re(S)\rangle = 1/g^2$. Inserting this into Eq. (2.5), restoring the Planck mass, and rescaling δ_{gs} and a by $1/g^2$ we have

$$\begin{aligned}\mathcal{L}_{eff} = &-M_p^2\partial_\mu\psi\partial^\mu\psi - (D_\mu\Phi_i)^\dagger D^\mu\Phi_i - \frac{e^{-2\psi}}{4g^2}F^{\mu\nu}F_{\mu\nu} + \frac{a}{4g^2M_p}F_{\mu\nu}\tilde{F}^{\mu\nu} \\ &-M_p^2 e^{4\psi}\left(\frac{\partial^\mu a}{2M_p} - \delta_{gs}A^\mu\right)^2 - \frac{g^2e^{2\psi}}{2}\left(\delta_{gs}M_p^2e^{2\psi} + X_i\Phi_i^\dagger\Phi_i\right)^2.\end{aligned} \tag{3.2}$$

This is invariant under local U(1) gauge transformations [with gauge parameter $\lambda(x^\mu)$] which now read

$$\Phi_i \to e^{iX_i\lambda}\Phi_i \quad , \quad A_\mu \to A_\mu + \partial_\mu\lambda \quad , \quad a \to a + 2M_p\delta_{gs}\lambda. \tag{3.3}$$

As discussed above, the gauge variation of the axion in the $F\tilde{F}$ term cancels the anomalous variation of the Lagrangian due to the (suppressed) fermions. In weakly coupled string theory, the anomaly coefficient δ_{gs} is calculated to be [5]

$$\delta_{gs} = \frac{1}{192\pi^2}\sum_i X_i, \tag{3.4}$$

where the sum is over the U(1) charges of the massless fermions and hence, by supersymmetry, over the charges of the massless bosons. In semi-realistic string models this sum may be large. A particular example furnished by the free-fermionic construction [13] yields $Tr(Q_X) = 72/\sqrt{3}$, so that $\delta_{gs} \sim 10^{-2}$. Assuming without loss of generality that $\delta_{gs} > 0$, the presence of a single scalar with negative charge can minimize the potential in Eq. (3.2) (assuming we assign the other scalars zero VEVs), thereby canceling the Fayet-Iliopoulos D-term, restoring supersymmetry, and spontaneously breaking the U(1) gauge symmetry. Thus, as in [12], we consider a single Higgs scalar Φ with negative unit charge, effectively ignoring quantum fluctuations of the other scalars about their zero VEVs, and working in the classical limit. This is consistent with ignoring the fermionic constributions.

Then Eq. (3.2) essentially becomes an Abelian Higgs model, coupled to the dilaton and axion through the anomaly, which may be viewed as a perturbation. To motivate this perspective, introduce a fictitious scaling parameter α so that

$$\delta_{gs} \to \alpha\delta_{gs}. \tag{3.5}$$

Then, as $\alpha \to 0$, the anomaly is turned off. In order for the spontaneously broken Abelian Higgs model to remain in this limit, the invariance of the term $\delta_{gs}M_p^2 e^{2\psi}$ in the potential, and in turn the gauge transformation of the axion, imply respectively that M_p and a should scale as :

$$M_p \to \alpha^{-1/2}M_p \quad , \quad a \to \alpha^{1/2}a. \tag{3.6}$$

Next we switch to dimensionless variables using the symmetry breaking scale defined by $\delta_{gs}^{1/2}M_p$[1]

$$\hat{x}^\mu = g\delta_{gs}^{1/2}M_p x^\mu \quad , \quad \hat{\phi} = \frac{\phi}{\delta_{gs}^{1/2}M_p} \quad , \quad \hat{A}^\mu = \frac{A^\mu}{g\delta_{gs}^{1/2}M_p} \quad , \quad \hat{a} = \frac{a}{\delta_{gs}M_p}, \tag{3.7}$$

where we have written $\Phi = \phi e^{i\eta}$, so $(D_\mu\Phi)^\dagger D^\mu\Phi = \partial_\mu\phi\,\partial^\mu\phi + \phi^2\left(\partial_\mu\eta + A_\mu\right)^2$. By design, these dimensionless variables are α invariants as required for a consistent perturbation scheme. Effecting these transformations and dropping the hats, we arrive at our final Lagrangian form:

$$\begin{aligned}\mathcal{L}'_{eff} = & \frac{-1}{\alpha\delta_{gs}}\partial_\mu\psi\partial^\mu\psi - \partial_\mu\phi\,\partial^\mu\phi - \phi^2(\partial_\mu\eta + A_\mu)^2 \\ & -\frac{e^{-2\psi}}{4}F^{\mu\nu}F_{\mu\nu} - \frac{e^{2\psi}}{2}\left(\phi^2 - e^{2\psi}\right)^2 \\ & +\alpha\delta_{gs}\left[\frac{a}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{e^{4\psi}}{4}\left(\partial^\mu a - 2A^\mu\right)^2\right],\end{aligned} \tag{3.8}$$

where we have rescaled the overall Lagrangian by the factor $M_p^4 g^2\delta_{gs}^2$. In the limit $\alpha\,\delta_{gs} \to 0$, we identically get the spontaneously broken Abelian Higgs model[2]. Thus, since only the combination $\alpha\delta_{gs}$ appears, setting $\alpha = 1$ (or relabeling $\beta = \alpha\delta_{gs}$), the only remaining parameter is δ_{gs} (or β) which is now to be interpreted as a perturbation

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[1]As typically $\delta_{gs}^{1/2} < 10^{-1}$, the tension of our vortex solutions, which is set by the scale of the spontaneous U(1) breaking, is below the Planck scale, justifying our neglect of metric back reaction in our analysis of these solutions.

[2]As we will later show explicitly, in this limit, the dilaton $\psi \to \langle\psi\rangle \equiv 0$, so its gradients vanish identially.

parameter[3].

The field equations derived from Eq. (3.8) are

$$\Box\psi = \frac{\beta}{2}\left[e^{2\psi}(3e^{2\psi}-\phi^2)(e^{2\psi}-\phi^2) - \frac{e^{-2\psi}}{2}F^{\mu\nu}F_{\mu\nu}\right] + \frac{\beta^2}{2}e^{4\psi}(\partial^\mu a - 2A^\mu)^2 \tag{3.9}$$

$$\Box\phi = \phi(\partial_\mu\eta + A_\mu)^2 + e^{2\psi}\phi(\phi^2 - e^{2\psi}) \tag{3.10}$$

$$0 = \partial_\mu[\phi^2(\partial^\mu\eta + A^\mu)] \tag{3.11}$$

$$\Box a = 2\partial_\mu A^\mu - \frac{e^{-4\psi}}{2}F_{\mu\nu}\tilde{F}^{\mu\nu} - 4\partial_\mu\psi(\partial^\mu a - 2A^\mu) \tag{3.12}$$

$$\partial_\mu(e^{-2\psi}F^{\mu\nu}) = 2\phi^2(\partial^\nu\eta + A^\nu) + \beta\left[\partial_\mu(a\tilde{F}^{\mu\nu}) - e^{4\psi}(\partial^\nu a - 2A^\nu)\right]. \tag{3.13}$$

First we note that despite the presence of the dynamical dilation, by differentiating Eq. (3.13) with respect to x^ν, and then using Eqs. (3.11), (3.12), and $\partial_\mu\tilde{F}^{\mu\nu} = 0$, we still obtain

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = 0. \tag{3.14}$$

Then, after choosing the Lorentz gauge $\partial_\mu A^\mu = 0$, the axion field equation (3.12) simplifies to

$$\Box a = -4\partial_\mu\psi(\partial^\mu a - 2A^\mu). \tag{3.15}$$

IV. VORTEX ODE'S

It is well known that the spontaneously broken Abelian Higgs model possesses topologically stable vortex solutions sometimes called Nielsen-Olesen vortices [10] (see Shellard and Vilenkin [11] for a complete reference on the subject). These correspond to static, cylindrically symmetrical solutions of the field equations for the Higgs and gauge fields. Specifically, working in cylindrical coordinates (t, r, θ, z) we look for solutions independent of t and z, with the standard vortex Ansatz [10], [11] for the Higgs phase and the gauge field:

$$\begin{aligned}\eta &= n\theta,\\ A^\mu &= (0, 0, A^\theta(r), 0) \equiv (0, 0, A(r), 0),\end{aligned} \tag{4.1}$$

where n is an integer characterizing the winding number of the vortex. The Higgs field $\Phi = \phi e^{i\eta} \to \langle\phi\rangle e^{i\eta}$ (as $r \to \infty$) defines a representation of the U(1) gauge group space S^1 since from Eq. (3.3), $\Phi \to e^{-i\lambda}\Phi$ under a gauge transformation. Thus Φ defines (as $r \to \infty$) a mapping from the boundary S^1 of physical space onto the group space S^1, and so can topologically be classified by an integer n. In the language of homotopy theory $\pi_1(S^1) = \mathcal{Z}$. With these Ansatze, the Higgs phase field equation (3.11) can be written as

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta}(\frac{n}{r} + A) = 0, \tag{4.2}$$

where we have used $\partial_\mu A^\mu = 0$ and the fact that $\eta = n\theta$ implies $\Box\eta = 0$. Then since in general $A(r) \neq -n/r$, we get

$$\frac{\partial\phi}{\partial\theta} = 0 \quad \Rightarrow \quad \phi = \phi(r). \tag{4.3}$$

This is normally assumed as an Ansatz, but this shows it actually follows from the Higgs phase field equation. Then Eq. (3.11) is identically satisfied with these forms of η, A, and ϕ. At this point we still have $a = a(r, \theta)$, and $\psi = \psi(r, \theta)$ assuming only static, axial symmetry. However, writing the Higgs modulus equation (3.10) as[4]

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[3]Strictly speaking, since the a defined here was rescaled by δ_{gs}, α is the perturbation parameter.
[4]Remember we are always working with metric signature $(-,+,+,+)$ so $\Box = -\frac{\partial^2}{\partial t^2} + \triangle$, etc.

$$
\begin{aligned}
\Box\phi - \phi(\partial_\mu\eta + A_\mu)^2 &= \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \phi(r)\left[\frac{n}{r} + A(r)\right]^2 \\
&\equiv f(r) \\
&= e^{2\psi(r,\theta)}\phi(r)\left[\phi^2(r) - e^{2\psi(r,\theta)}\right] \qquad (4.4)
\end{aligned}
$$

determines ψ algebraically as a function of r alone, so $\psi = \psi(r)$. Furthermore, consider the gauge field equation (3.13) for $\nu = r$, i.e. $\nu = 1$. Since $A^\mu = \delta^{2\mu}A(r)$, only F^{12} and $\tilde{F}^{03}$ are nonzero. Then Eq. (3.13) for $\nu = 1$ reads

$$
\frac{1}{r}\frac{\partial}{\partial\theta}\left[e^{-2\psi(r)}F^{21}(r)\right] \equiv 0 = 2\phi^2(0+0) + \beta\left[0 - e^{4\psi}(\frac{\partial a}{\partial r} - 0)\right] \qquad \Rightarrow \qquad \frac{\partial a}{\partial r} = 0, \qquad (4.5)
$$

so that $a = a(\theta)$. Now $\psi = \psi(r)$, $a = a(\theta)$, and $A = A(r)$ imply in the axion field equation (3.15) that

$$
\partial_\mu\psi(\partial^\mu a - 2A^\mu) = 0 \qquad \Rightarrow \qquad \Box a = \frac{1}{r^2}\frac{d^2a}{d\theta^2} = 0. \qquad (4.6)
$$

This fixes

$$
a(\theta) = C\theta + D. \qquad (4.7)
$$

Because a appears only derivatively coupled, we may take without loss of generality $D = 0$. Furthermore, single valuedness in the physical space requires that C be an integer, so that a represents a mapping from physical space into the gauge group space just as η does (see [12]). The specific axion solution of Binétruy, Deffayet and Peter [12] corresponds to the choice $C = -2n$, where n is the winding number of the Higgs phase[5]. We will consider the general case for the moment, leaving $C = -2m$ without loss of generality (m integral or half-integral), with m not necessarily equal to n. Effectively this allows the axion and the Higgs phase to have different winding numbers.

Combining what we have learned about the coordinate dependences of the fields, we can now reduce the remaining field equations (3.9), (3.10), and (3.13) to three ordinary differential equations:

$$
\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = \frac{\beta}{2}\left[e^{2\psi}(3e^{2\psi} - \phi^2)(e^{2\psi} - \phi^2) - e^{-2\psi}\left(\frac{1}{r}\frac{d}{dr}(rA)\right)^2\right] + 2\beta^2 e^{4\psi}\left(\frac{m}{r} + A\right)^2, \qquad (4.8)
$$

$$
\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = \phi\left(\frac{n}{r} + A\right)^2 + e^{2\psi}\phi\left(\phi^2 - e^{2\psi}\right), \qquad (4.9)
$$

$$
\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rA)\right] = 2\frac{d\psi}{dr}\frac{1}{r}\frac{d}{dr}(rA) + 2\phi^2 e^{2\psi}\left(\frac{n}{r} + A\right) + 2\beta e^{6\psi}\left(\frac{m}{r} + A\right). \qquad (4.10)
$$

As in the standard Nielsen-Olesen vortices [10], [11] of the Abelian Higgs model, we require that the Higgs modulus approach its vacuum expectation value asymptotically to minimize the potential term, and that the covariant derivative $D_\mu\Phi$ vanish asymptotically (i.e. the gauge field asymptotically becomes a pure gauge) so that the energy (per unit length) of the vortex remains finite. Translated into our language, these conditions read:

$$
\begin{aligned}
&\phi(r) \to 1 \quad , \quad r \to \infty; \\
&A(r) \to \frac{-n}{r} \quad , \quad r \to \infty. \qquad (4.11)
\end{aligned}
$$

The Higgs 'screening' by the gauge fields prevents the logarithmic divergence of global vortices, so that the energy integral $\int(\frac{n}{r} + A)^2\phi^2 r dr$ (remnants of the covariant derivative $D_\mu\Phi$) is asymptotically finite. However, after fixing the asymptotic gauge behavior with respect to the Higgs boson, the presence of the axion kinetic term $\int(\frac{m}{r} + A)^2 r dr$ reintroduces these logarithmic divergences in the energy integral, unless $m = n$ (the result of Binétruy et al.). Since

[5] $a = -2n\theta$ in the original variables reads $a = 2\delta_{gs}M_p\eta/X$

our primary interest is now in the dilaton, for the remainder of our discussion we consider the $m = n$ case to simplify the equations slightly. We demonstrate in the next section that this will in no way affect any subsequent results.

Before proceeding we now make a convenient change of variables for the gauge field. Define $v(r)$ through

$$A(r) = \frac{-n[1 - v(r)]}{r}, \tag{4.12}$$

so that

$$v(r) \to 0 \quad , \quad r \to \infty. \tag{4.13}$$

Equations (4.8)-(4.10) now read, denoting r derivatives by primes

$$\psi'' + \frac{\psi'}{r} = \frac{\beta}{2}\left[3e^{6\psi} - 4\phi^2 e^{4\psi} + \phi^4 e^{2\psi} - \frac{e^{-2\psi}n^2}{r^2}(v')^2\right] + 2\beta^2 e^{4\psi}\frac{n^2 v^2}{r^2}, \tag{4.14}$$

$$\phi'' + \frac{\phi'}{r} = \frac{n^2}{r^2}\phi v^2 + e^{2\psi}\phi(\phi^2 - e^{2\psi}), \tag{4.15}$$

$$v'' - \frac{v'}{r} = 2\psi' v' + 2(\phi^2 e^{2\psi} + \beta e^{6\psi})v. \tag{4.16}$$

We require the dilaton to approach its asymptotic VEV as $r \to \infty$, which, in our langauge, means

$$\psi \to 0 \quad , \quad r \to \infty \quad (i.e.\langle Re(S)\rangle = \frac{1}{g^2}). \tag{4.17}$$

Now consider the boundary conditions at $r = 0$. In the standard Nielsen-Olesen or Abelian Higgs model [11], the vortex configuration means that ϕ attains the symmetric (false vacuum) state $\phi = 0$ at $r = 0$ (which we argued was necessary for the energy integral to be well defined), and A remains bounded (more precisely the magnetic field remains bounded). Thus we have

$$\phi(0) = 0 \quad , \quad v(0) = 1. \tag{4.18}$$

This leaves, finally, the boundary condition for the dilaton at $r = 0$. Of course we would like to have the dilaton (VEV) remain bounded in the core, but as we shall now show, this is not possible if $\beta \neq 0$.

V. PERTURBATIVE EXPANSION AND CORRECTIONS TO THE DILATON

Throughout this section we will make usage of the following elementary fact of our radial equations:

$$f'' + \frac{f'}{r} = 0 \quad \Rightarrow f(r) = C_1 + C_2 \log(r). \tag{5.1}$$

First, note that if $\beta = 0$, then the dilaton equation (4.14) becomes Eq. (5.1), so that the asymptotic condition (4.17) on the dilaton then implies:

$$\psi_0(r) \equiv 0 \quad \forall r. \tag{5.2}$$

This of course corresponds to the frozen dilaton. Then the other two equations, Eqs. (4.15) and (4.16), identically reduce to the Nielsen-Olesen equations of the Abelian Higgs model, as promised:

$$\phi_0'' + \frac{\phi_0'}{r} = \frac{n^2}{r^2}\phi_0\, v_0^2 + \phi_0\left(\phi_0^2 - 1\right), \tag{5.3}$$

$$v_0'' - \frac{v_0'}{r} = 2\phi_0^2\, v_0, \tag{5.4}$$

with $v_0(0) = 1$, $v_0(\infty) = 0$, $\phi_0(0) = 0$, $\phi_0(\infty) = 1$. We have subscripted the fields with zeros to indicate that these are the zeroth order terms in a perturbation expansion in β, which we now define formally in the obvious way:

$$\psi(r)=\sum_{i=0}^{\infty}\beta^i\psi_i(r) \quad , \quad \phi(r)=\sum_{i=0}^{\infty}\beta^i\phi_i(r) \quad , \quad v(r)=\sum_{i=0}^{\infty}\beta^i v_i(r). \tag{5.5}$$

Substituting these into Eqs. (4.14)-(4.16) yields the following $O(\beta)$ corrections:

$$\psi_1''+\frac{\psi_1'}{r}=\frac{1}{2}\left[3-4\phi_0^2+\phi_0^4-\frac{n^2}{r^2}(v_0')^2\right], \tag{5.6}$$

$$\phi_1''+\frac{\phi_1'}{r}=\frac{n^2}{r^2}\left(\phi_1 v_0^2+2\phi_0 v_0 v_1\right)+2\psi_1\left(\phi_0^3-2\phi_0\right)+\phi_1\left(3\phi_0^2-1\right), \tag{5.7}$$

$$v_1''-\frac{v_1'}{r}=2\psi_1' v_0'+2v_0\left(2\phi_0\phi_1+2\phi_0^2\psi_1+1\right)+2v_1\phi_0^2, \tag{5.8}$$

where we have included the corrections to the Higgs and gauge field for completeness. What really interests us is the first of these equations, Eq. (5.6), the first correction to the dilaton. Note that this $O(\beta)$ correction does *not* depend on having chosen the choice of Binétruy et al. for the axion, since the axion does not enter at this order. This can be seen directly from Eq. (3.8) or (4.14). More importantly, this dilaton correction can be calculated from knowledge of only ϕ_0 and v_0, i.e. the Nielsen-Olesen solution for the Higgs and the gauge field.[6]

Unfortunately explicit solutions to the Nielsen-Olesen equations (5.3)-(5.4) are not known. However, all we really need is a parametrization of the solutions with the correct behavior at $r\to\infty$ and at $r\to 0$. The conclusions we will draw, will depend only on the asymptotic behavior of ϕ_0, v_0, *and* in particular the $r\to\infty$ boundary condition on ψ itself.

Thus, first consider the large r behavior of the Nielsen-Olesen equations (5.3), (5.4). Write ϕ_0 and v_0 as $1-\delta\phi_0$ and δv_0 respectively, where δ's represent deviations with respect to asymptotic values. Then the linearizations of Eqs. (5.3),(5.4) are

$$\delta\phi_0''+\frac{\delta\phi_0'}{r}=2\delta\phi_0+O(\delta^2), \tag{5.9}$$

$$\delta v_0''-\frac{\delta v_0'}{r}=2\delta v_0+O(\delta^2). \tag{5.10}$$

Note that as per Perivolaropoulos [14] (or Shellard and Vilenkin [11]), since we have the case '$\beta<4$' (in their notation), we do not need to consider the inhomogeneous term $(\delta v_0)^2/r^2$ in the $\delta\phi_0$ equation, which can dominate a linear term of $O(\delta\phi_0)$ if $\beta>4$. In this case, the gauge field dictates the falloff of the Higgs field. Our 'β' (not to be confused with the perturbation parameter) is 1, so this usual (strict) linearization applies. The solutions to these linearized equations, with the asymptotic boundary conditions, are in terms of modified Bessel functions:

$$\delta\phi_0\to K_0(\sqrt{2}r)\to C_\phi\frac{e^{-\sqrt{2}r}}{\sqrt{r}} \quad , \quad r\to\infty, \tag{5.11}$$

$$\delta v_0\to K_1(\sqrt{2}r)\to C_v\sqrt{r}e^{-\sqrt{2}r} \quad , \quad r\to\infty, \tag{5.12}$$

where C_ϕ, and C_v are constants of order 1. As Perivolaropolous [14] notes, the factor of $1/\sqrt{r}$ is usually neglected in Eq. (5.11). We will neglect these $\sqrt{r}$ terms as being negligible with respect to the exponentials when parametrizing a solution of the Nielsen-Olesen equations over the whole range, and later argue that this does not affect our results.

Now consider the small r behavior, this time taking ϕ_0 as $\delta\phi_0$. With $v_0(r\ll 1)\approx 1$ the leading order behavior of Eq. (5.3) at small r is

$$\delta\phi_0''+\frac{\delta\phi_0'}{r}=\frac{n^2\delta\phi_0}{r^2} \quad \Rightarrow \quad \delta\phi_0=Ar^n \quad , \quad r\ll 1 \tag{5.13}$$

where $A>0$ (to be determined conveniently in a moment), and where we have discarded the second singular solution. At this point we specialize to the $n=\pm 1$ vortex for simplicity. Then the small r gauge field equation is

[6] In fact, it is obvious that the dilaton at any order is determined only by functions of lower order.

$$v_0'' - \frac{\delta v_0'}{r} = 2(\delta\phi_0)^2 v_0 = 2A^2 r^2 v_0, \tag{5.14}$$

with solution

$$v_0 = e^{-Ar^2/\sqrt{2}} \sim 1 - \frac{A}{\sqrt{2}} r^2 + O(r^4) \quad , \quad r \ll 1, \tag{5.15}$$

where again we have discarded the second solution (a positive exponential), which has the wrong behavior near $r = 0$, and used $v_0(0) = 1$. Combining Eqs. (5.11), (5.12), (5.13), and (5.15) suggests the following parametrizations of the solutions to the Nielsen-Olesen equations:

$$\phi_0(r) \sim \tanh(\frac{r}{\sqrt{2}}), \tag{5.16}$$

$$v_0(r) \sim \mathrm{sech}^2(\frac{r}{\sqrt{2}}), \tag{5.17}$$

which corresponds to setting $A = 1/\sqrt{2}$. They have the following asymptotic behavior:

$$\phi_0(r) \to \frac{r}{\sqrt{2}} \quad (r \to 0) \quad ; \quad \phi_0(r) \to 1 - 2e^{-\sqrt{2}r} \quad (r \to \infty) \tag{5.18}$$

$$v_0(r) \to 1 - \frac{r^2}{2} \quad (r \to 0) \quad ; \quad v_0(r) \to 4e^{-\sqrt{2}r} \quad (r \to \infty), \tag{5.19}$$

and are therefore suitable parametrizations that become ‘exact’ in both r limits.[7] These are of course the usual solitonic-type forms that qualitatively describe the behavior of the solutions to Eqs. (5.3),(5.4) very well, as can be checked by comparing them with the exact numerical calculations.

Inserting Eqs. (5.16) and (5.17) into the dilaton correction (5.6) yields, after some trigonometric simplifcation,

$$\psi_1'' + \frac{\psi_1'}{r} = \mathrm{sech}^2(\frac{r}{\sqrt{2}}) + \mathrm{sech}^4(\frac{r}{\sqrt{2}}) \frac{\left[\mathrm{sech}^2(\frac{r}{\sqrt{2}}) - \left(1 - \frac{r^2}{2}\right)\right]}{r^2} \equiv f(r). \tag{5.20}$$

However, the inhomogeneous right hand side is well approximated globally by the first term $\mathrm{sech}^2(r/\sqrt{2})$. In particular, the dominant asymptotic behavior as $r \to \infty$ is the same [since the latter term is a correction of $O(\exp(-2\sqrt{2}r))$ coming from the $(v_0')^2$ and the ϕ_0^4 contributions], and is correct to $O(r)$ in the small r limit.[8] Thus we take

$$\psi_1'' + \frac{\psi_1'}{r} \simeq \mathrm{sech}^2(\frac{r}{\sqrt{2}}) \quad (\to 4e^{-\sqrt{2}r} \text{ as } r \to \infty), \tag{5.21}$$

where we have included the explicit asymptotic behavior for later usage. The general solution of Eq. (5.21) is a particular solution of the inhomogeneous equation, plus the fundamental solution (5.1) with the arbitrary constants chosen to satisfy the boundary conditions. The general solution for $\psi_1(r)$,

$$\psi_1(r) = \log(r) \int r \ \mathrm{sech}^2(\frac{r}{\sqrt{2}}) dr - \int r \log(r) \mathrm{sech}^2(\frac{r}{\sqrt{2}}) dr + C_1 + C_2 \log(r), \tag{5.22}$$

with the requirement that $\psi_1(\infty) = 0$. Evaluating the first integral explicitly, and then integrating the second integral by parts using the result just obtained, allows us to bring this to the much more convenient form,

$$\psi_1(r) = \int_a^r \left[\sqrt{2} \tanh(\frac{x}{\sqrt{2}}) - 2 \frac{\log[\cosh(\frac{x}{\sqrt{2}})]}{x} \right] dx + C_1 + C_2 \log r, \tag{5.23}$$

[7] A quick numerical check reveals that the error, by construction, is concentrated near $r = 1$ and is bounded above by about 20%.

[8] Alternatively, we do not have to make this truncation, at the price of making the subsequent analysis much more algebraically tedious, without qualitatively changing the result. The point is that it will be the dominant asymptotic behavior that determines the dilaton behavior.

where we have introduced a lower integration limit a, to be determined momentarily. In order to be able to impose the boundary condition $\psi_1(\infty) = 0$, we need to understand the convergence of this integral as a (type I) improper integral. It is easy to show that in fact the integral is logarithmically divergent as $r \to \infty$ since,

$$\lim_{r\to\infty} \frac{\sqrt{2}\tanh(\frac{r}{\sqrt{2}}) - 2\frac{\log[\cosh(\frac{r}{\sqrt{2}})]}{r}}{\frac{1}{r}} = 2\log(2). \tag{5.24}$$

If we rewrite the integrand in terms of exponentials, this limit is made more evident, as well as allowing us to write a closed form expression for the integral. Denoting the integrand by $F(r)$ we have

$$\begin{aligned} F(r) &= \sqrt{2}\left[\frac{1-e^{-\sqrt{2}r}}{1+e^{-\sqrt{2}r}}\right] - \sqrt{2} - \frac{2\log(1+e^{-\sqrt{2}r})}{r} + \frac{2\log(2)}{r} \\ &= 2\sqrt{2}\sum_{n=1}^{\infty}(-1)^n e^{-n\sqrt{2}r} - \frac{2}{r}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-n\sqrt{2}r}}{n} + \frac{2\log(2)}{r}, \end{aligned} \tag{5.25}$$

whence it is clear that the last term yields the logarithmic divergence, whereas the other terms yield obviously convergent integrals. This divergence must be canceled by the $C_2\log(r)$ term of the homogeneous solution (5.1), by setting $C_2 = -2\log(2)$. This is a necessary condition of being able to impose $\psi_1(\infty) = 0$. Then, pulling the homogeneous solution $-2\log(2)\log(r)$ under the integral to cancel the $2\log(2)/r$ piece, to fully impose the boundary condition we must take the integration limit a to infinity since the integrand is monotonic. Also, we must take the constant homogeneous solution $C_1 = 0$. Putting it all together, we finally have

$$\begin{aligned} \psi_1(r) &= \int_{\infty}^{r}\left[2\sqrt{2}\sum_{n=1}^{\infty}(-1)^n e^{-n\sqrt{2}r} - \frac{2}{r}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-n\sqrt{2}r}}{n}\right]dr \\ &= 2\log(1+e^{-\sqrt{2}r}) + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Ei}_1(n\sqrt{2}r), \end{aligned} \tag{5.26}$$

where we have introduced the exponential integral defined by

$$\mathrm{Ei}_1(x) = \int_1^{\infty}\frac{e^{-xt}}{t}dt. \tag{5.27}$$

It is easy to verify explicitly that this solves the dilaton correction equation (5.21) and satisfies

$$\lim_{r\to\infty}\psi_1(r) = 0. \tag{5.28}$$

However, though we have been able set the dilaton ψ equal to zero at spatial infinity, the dilaton now diverges to $+\infty$ at $r = 0$ since

$$\begin{aligned} \lim_{r\to 0}\psi_1(r) = \lim_{r\to 0} 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Ei}_1(n\sqrt{2}r) &\sim \lim_{r\to 0} -2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\log(r) \\ &= \lim_{r\to 0} -2\log(2)\log(r) \to +\infty, \end{aligned} \tag{5.29}$$

using the fact that

$$\lim_{r\to 0}\frac{\mathrm{Ei}_1(ar)}{-\log(r)} = \lim_{r\to 0}\frac{\frac{-e^{-ar}}{r}}{\frac{-1}{r}} = 1 \quad \forall a > 0. \tag{5.30}$$

How did this come about? This singularity is none other than the one introduced when we were forced to assign a nonzero value to the homogeneous term $C_2\log(r)$ in order to obey the boundary condition at infinity. Thus in order to avoid a logarithmic divergence at infinity, we are forced to introduce one at zero by turning on $\log(r)$. This can be viewed as a direct consequence of the fact that we are dealing with an essentially two-dimensional problem and the two-dimensional Laplace equation.

It is now clear why this result is independent of the parametrizations (5.16),(5.17), and of the truncation made in going to Eq. (5.21). The $C_2 \log(r)$ homogeneous term is turned on (and effectively shifts the particular solution) if and only if the (unshifted) particular solution integral is asymptotically divergent, which in turn depends only on the dominant asymptotic behavior of the Nielsen-Olesen solutions. But this is precisely how we chose the parametrization and made the truncation: they have the correct asymptotic behavior. Conversely, once the $C_2 \log(r)$ term is turned on, we now unavoidably have a positive logarithmic divergence at $r = 0$, because the *unshifted* integrand is well behaved near $r = 0$. Again, we chose our parametrization to have the correct small r behavior of the Nielsen-Olesen solutions.

Finally one might worry in taking, as most authors including Nielsen and Olesen do, the asymptotic behavior of ϕ_0 as $\exp(-\sqrt{2}r)$ and not $\exp(-\sqrt{2}r)/\sqrt{r}$, that we may have affected the convergence of the unshifted particular integral. This is not the case. Proceeding exactly as above, and retaining only the dominant asymptotic contribution, it is easy to show that the boundary condition at infinity again forces us to turn on the homogeneous solution. We worked with a simpler global parametrization before, so that we could discuss small r behavior of the solution as well.

VI. DISCUSSION

The results of the previous section are perhaps surprising. In fact, this is a rather generic property of solutions to the inhomogeneous equation

$$\psi_1'' + \frac{\psi_1'}{r} = f(r) \tag{6.1}$$

with a vanishing asymptotic boundary condition, and with reasonable assumptions on $f(r)$. As we have seen, the general solution of Eq. (6.1) can be written as

$$\begin{aligned}\psi_1(r) &= \log(r) \int r f(r) dr - \int r \log(r) f(r) dr + C_1 + C_2 \log(r) \\ &= \log(r) \int_a^r x f(x) dx - \int_b^r x \log(x) f(x) dx,\end{aligned} \tag{6.2}$$

where we have absorbed the homogeneous solution into the particular indefinite integrals by making them definite integrals: the arbitrary constants of the general solution are now the lower, constant, limits of integration. Clearly, we cannot in general impose the boundary condition $\psi_1(\infty) = 0$. A necessary condition for being able to impose this condition is that

$$\lim_{r\to\infty} \int^r x \log(x) f(x) dx \tag{6.3}$$

exists. Unfortunately, this is not quite sufficient ($f(x) = \sin(x^2)/[x \log(x)]$ furnishes a counterexample). However, the *absolute* convergence of the integral (6.3) is sufficient to be able to impose $\psi_1(\infty) = 0$, i.e. if

$$\lim_{r\to\infty} \int^r x \log(x) \left|f(x)\right| dx = K < \infty. \tag{6.4}$$

For if this limit exists, then so does the limit

$$\lim_{r\to\infty} \int^r x \left|f(x)\right| dx. \tag{6.5}$$

Then the squeeze theorem and the inequalities

$$0 \le \left|\log(r) \int_r^\infty x f(x) dx\right| \le \int_r^\infty \log(r) x \left|f(x)\right| dx \le \int_r^\infty \log(x) x \left|f(x)\right| dx \to 0 \ \ as \ \ r \to \infty \tag{6.6}$$

imply that

$$\lim_{r\to\infty} \log(r) \int_r^\infty x f(x) dx = 0. \tag{6.7}$$

This establishes the sufficiency of the condition (6.4).

From Eq. (5.6), the actual $f(r)$ in which we are interested is determined from the Nielsen-Olesen solutions ϕ_0 and v_0, and the arguments from the previous section establish that this $f(r)$ decays exponentially as $r \to \infty$. Thus we easily satisfy the above sufficient condition allowing us to take $\psi_1(\infty) = 0$.

Now consider the behavior of $\psi_1(r)$ near $r = 0$, *subsequent* to imposing $\psi_1(\infty) = 0$. We now write the solution (6.2) as

$$\psi_1(r) = \int_r^\infty x \log(x) f(x) dx - \log(r) \int_r^\infty x f(x) dx. \tag{6.8}$$

Remembering that $x \log(x) \to 0$ as $x \to 0^+$, we now demonstrate the inevitable presence of a logarithmic divergence of $\psi_1(r)$ at $r = 0$ as long as $f(r)$ is well behaved near $r = 0$ and $K \equiv \int_0^\infty x f(x) dx \neq 0$. The sign of the divergence will depend on the sign of K. Explicitly we have

$$\lim_{r \to 0} \psi_1(r) \sim \int_0^\infty x \log(x) f(x) dx - \log(r) \int_0^\infty x f(x) dx \to \mathrm{sgn}(K) \cdot \infty. \tag{6.9}$$

Note that these integrals exist assuming only, in addition to the previous restrictions on f ensuring improper convergence, that f is defined and say continuous (or Riemann integrable) everywhere on $r \geq 0$, and in particular at 0. [9]

Again, because our $f(r)$ from Eq. (5.6) is defined and continuous for all $r \geq 0$ because the Nielsen-Olesen solutions are [remember that the term $(v_0')^2/r^2$ in Eq. (5.6) is finite as $r \to 0$ as seen in Eq. (5.20); in other words the field strength of the Nielsen-Olesen vortex is finite at the core], we have a logarithmic divergence at $r = 0$ as explicitly shown in the previous section. In fact, since our $f(r)$ is explicitly non-negative (as seen in either Eq. (5.20) or its truncation (5.21)), the K defined above is positive, and so the logarithmic divergence is to *positive* infinity at $r = 0$. Again, this was seen explicitly in the last section.

To summarize, we have found that a solution to Eq. (6.1) can satisfy $\psi_1(\infty) = 0$, if the limit (6.4) exists. Furthermore, if this limit exists so that we may impose $\psi_1(\infty) = 0$, the solution diverges logarthmically at $r = 0$. Thus $\psi(\infty) = 0$ implies $\psi(0) = \infty$. Since the $f(r)$ relevant to our discussion decays exponentially as $r \to \infty$, and is well behaved at $r = 0$, this is provides a general and generic proof of our result. Incidentally, this also shows why our results of the previous section are independent of either the parametrizations to the Nielsen-Olesen solutions or the truncation made in going from Eq. (5.20) to Eq. (5.21): this general behavior depends only on the behavior of f as $r \to \infty$ and as $r \to 0$, and our parametrization was chosen to be exact in these limits.

Given that we have now established that this dilaton behavior is rather generic, one might wonder if this divergent behavior of the dilaton at the core of the vortex is somehow an artifact of the perturbation theory. In fact, we now expect the full dilaton equation to yield even worse behavior because of the exponential feedback. As a consistency check of our result, we will briefly examine the full dilaton equation (4.14). If we take the perturbation theory to be valid only for very large r, where the dilaton VEV is still small, so that we are still in a classical and perturbative regime, we know that it starts to run positive as one comes in from spatial infinity. A positive exponential self-coupling acts as a source term that becomes larger and larger as $r \to 0$. So if we equate small r with large ψ, then the dilaton equation (4.14) is dominated by the vacuum Fayet-Iliopoulos term [2] proportional to $e^{6\psi}$ [or $1/(S + S^\dagger)^3$ in the notation of Polchinski], which comes directly from the anomaly cancellation as a two string-loop tadpole [5], so that, approximately

$$\psi'' + \frac{\psi'}{r} \sim \frac{3\beta}{2} e^{6\psi}, \tag{6.10}$$

where we are taking β so small that we can neglect the axion contribution that is otherwise possibly as large (but of the same sign in any case), and where we are assuming that we still have $\phi \to 0$ as $r \to 0$; i.e. the vortex is well defined. An exact solution to Eq. (6.10) is given by

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[9] Of course if f is poorly behaved (say divergent) as $r \to 0$, so that the integral diverges, then already the dilaton diverges without further argument.

$$\psi(r) \sim \frac{-1}{6} \log \left[a_1 r \left(1 - \frac{9\beta}{2a_1} r \right)^2 \right], \tag{6.11}$$

where a_1 is an undetermined constant. For very small β this is essentially the same behavior as our perturbative calculations. This solution is obviously consistent with the approximation (6.10) to the full dilaton equation (4.14) if we assume that the gauge field and Higgs boson still have the boundary values $\phi(0) = 0$, and $v'(0) = 0$.

In any case, we seem to be led to the conclusion that the 4-dimensional dilaton in this model starts to grow as we come in from spatial infinity. Since the dilaton VEV in this model sets the anomalous U(1) gauge coupling, we eventually enter a strongly coupled regime where not only the β perturbation theory breaks down, but where it no longer makes sense to ignore quantum and string threshold corrections. In other words, such a vortex is fundamentally a quantum mechanical object. Furthermore, as we have seen, the unavoidable singularities we have encountered are a direct consequence of the effectively *two*-dimensional nature of the vortex system: the solution of the Laplace (or Poisson) equation in two dimensions involves a logarithm which is singular at both $r = 0$ and $r \to \infty$.

Our conclusion then is that anomalous U(1) vortex solutions of heterotic superstring theory, if they are to have the standard asymptotic structure at large radial distances from the vortex core, necessarily generate large dilaton field values within that core signaling the presence of strong coupling and large quantum fluctuations. As such, these vortices can never be adequately described as entirely classical objects; their classical exterior surrounds an interior that is intrinsically quantum mechanical.

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